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What Makes it Special? : Content Knowledge for Teaching

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A B S T R A C T

This article reports the authors' efforts to develop a practice-based theory of content knowledge for teaching built on Shulman's (1986) notion of pedagogical content knowledge. As the concept of pedagogical content knowledge caught on, it was in need of theoretical development, analytic clarification, and empirical testing. The purpose of the study was to investigate the nature of professionally oriented subject matter knowledge in mathematics by studying actual mathematics teaching and identifying math ematical knowledge for teaching based on analyses of the mathematical problems that arise in teaching. In conjunction, mea sures of mathematical knowledge for teaching were developed. These lines of research indicate at least two empirically discernable subdomains within pedagogical content knowledge (knowledge of content and students and knowledge of content and teaching) and an important subdomain of "pure" content knowledge unique to the work of teaching, specialized content knowledge, which is distinct from the common content knowledge needed by teachers and nonteachers alike. The article con cludes with a discussion of the next steps needed to develop a useful theory of content knowledge for teaching.

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INTRODUCTION

Most people would agree that an understanding of content matters for teaching. Yet, what constitutes understanding of the content is only loosely defined. In the mid-1980s, a major breakthrough initiated a new wave of interest in the conceptualization of teacher con- tent knowledge. Lee Shulman (1986) and his colleagues proposed a special domain of teacher knowledge that they termed pedagogical content knowledge. What pro- voked broad interest was the suggestion that there is content knowledge unique to teaching a kind of subject matter specific professional knowledge. The continuing appeal of the notion of pedagogical content knowledge is that it bridges content knowledge and the practice of teaching. However, after two decades of work, this bridge between knowledge and practice was still inadequately understood and the coherent theoretical framework Shulman (1986, p. 9) called for remained knowledge is underspecified. The term has lacked defin ition and empirical foundation, limiting its usefulness.

Throughout the past 20 years, for example, researchers have used pedagogical content knowledge to refer to a wide range of aspects of subject matter knowledge and the teaching of subject matter and, indeed, have used it differently across and even within subject areas. Besides differences in the breadth of what the term includes, there have been significant differences in how the term is used to relate content knowledge to the prac- tice of teaching. Frequent, for example, are broad claims about what teachers need to know. Such statements are often more normative than empirical. Only a few studies have tested whether there are, indeed, distinct bodies of identifiable content knowledge that matter for teaching underdeveloped. This article builds on the promise of pedagogical content knowledge, reporting new progress on the nature of content knowledge for teaching.

For the last 15 years, the work of the Mathematics Teaching and Learning to Teach Project and of the Learning Mathematics for Teaching Project has focused both on the teaching of mathematics and on the mathe atics used in teaching. Although the context of our work has been mathematics, we have sought to contribute to a broader discussion by researchers in different school subjects. To consider the knowledge that teaching entails, we began by investigating what teaching itself demands. Instead of reasoning from the school curriculum to a list of topics teachers must know, we developed an empirical approach to understanding the content knowledge needed for teaching. The first project focused on the work teachers do in teaching mathematics. The authors and their colleagues used studies of teaching practice to analyze the mathematical demands of teaching and, based on these analyses, developed a set of testable hypotheses about the nature of mathematical knowledge for teaching. In a related line of work, the second project developed survey measures of content knowledge for teaching mathematics. The measures provided a way to investigate the nature, the role, and the importance of different types of mathematical knowledge for teaching.

In particular, these studies have led us to hypothesize some refinements to the popular concept of pedagogical content knowledge and to the broader concept of content knowledge for teaching. In this article, we focus on the work of teaching in order to frame our conceptualization of the mathematical knowledge and skill needed by teachers. We identify and define two empirically detectable subdomains of pedagogical content knowledge. In addi tion, and to our surprise, we have begun to uncover and articulate a less recognized domain of content knowledge for teaching that is not contained in pedagogical content knowledge, but yet we hypothesizeis essential to effective teaching. We refer to this as specialized content knowledge. These possible refinements to the map of teacher content knowledge are the subject of this article. Because the work of Shulman and his colleagues is foundational, we begin by reviewing the problem they framed, the progress they made, and the questions that remained unanswered. We use this discussion to clarify the problems of definition, empirical basis, and practical utility that our work addresses. We then turn to mathematics in particular, describe work on the problem of identifying mathematical knowledge for teaching, and report on refinements to the categories of mathematical knowledge for teaching. The article concludes with an appraisal of next steps in developing a useful theory of content knowledge for teaching.

LITERATURE REVIEW

Content Knowledge and Its Role in Defining Teaching as a Profession

A central contribution of Shulman and his colleagues was to reframe the study of teacher knowledge in ways that attend to the role of content in teaching. This was a radical departure from research of the day, which focused almost exclusively on general aspects of teaching. Subject matter was little more than context. Although earlier studies were conducted in classrooms where mathematics, reading, or other subjects were taught, attention to the subject itself and to the role it played in teaching or teacher thinking was less prominent. In fact, so little attention was devoted to examining content and its role in instruction that Shulman dubbed this the "missing paradigm" in research on teaching and teacher knowledge (1986).

A second contribution of Shulman and his colleagues was to represent content understanding as a special kind of technical knowledge key to the profession of teaching. In the late 1980s, they conducted case studies of beginning high school teachers as part of their research in the Knowledge Growth in Teaching project. Participants were recent graduates with strong subject matter preparation in mathematics, science, English literature, and history. By examining these novices in the process of learning to teach, the group sought to investigate how strong subject matter preparation translated into the knowledge needed for teaching that subject. Deliberately working across subjects provided a comparative basis for examining more general characteristics of the knowledge that the teachers used in their practice.

A closely related purpose was to draw from these cat- egories of teacher knowledge to inform the development of a National Board system for the certification of teachers that would "focus upon the teacher's ability to reason about teaching and to teach specific topics, and to base his or her actions on premises that can bear the scrutiny of the professional community" (Shulman, 1987, p. 20). Attention to certification was deliberately geared toward informing debates about what constitutes professional expertise and what such expertise implies.

Figure 1

Shulman's Major Categories of Teacher Knowledge

The remaining three categories define content-specific dimensions and together comprise what Shulman referred to as the missing paradigm in research on teaching"a blind spot with respect to content that characterizes mostresearch on teaching, and as a consequence, most of our state-level programs of teacher evaluation and teacher certification" (1986, pp. 7-8). The first, *content knowledge*, includes knowledge of the subject and its organizing structures (see also Grossman, Wilson, & Shulman, 1989; Wilson, Shulman, & Richert, 1987).

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter
- Knowledge of learners and their characteristics
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures
- Knowledge of educational ends, purposes, and values, and theirphilosophical and historical grounds
- Content knowledge
- Curriculum knowledge, with particular grasp of the materials and programsthat serve as "tools of the trade" for teachers
- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding

(Shulman, 1987, p. 8)

Drawing on Schwab (1961/1978), Shulman (1986) argued that know- ing a subject for teaching requires more than knowing its facts and concepts. Teachers must also understand the organizing principles and structures and the rules for establishing what is legitimate to do and say in a field. The teacher need not only understand that something is so; the teacher must further understand why it is so, on what for teacher preparation and for policy decisions. In particular, Shulman was concerned with prevailing concep- tions of teacher competency, which focused on generic teaching behaviors. He argued that "the currently incom- plete and trivial definitions of teaching held by the pol- icy community comprise a far greater danger to good education than does a more serious attempt to formulate the knowledge base" (Shulman, 1987, p. 20). Implicit in such comments is the argument that high-quality instruc- tion requires a sophisticated, professional knowledge that goes beyond simple rules such as how long to wait for students to respond.

To characterize professional knowledge for teaching, Shulman and his colleagues developed typologies. Although the specific boundaries and names of categories varied across publications, one of the more com- plete articulations is reproduced in Figure 1.

These categories were intended to highlight the important role of content knowledge and to situate con- tent-based knowledge in the larger landscape of profes- sional knowledge for teaching. The first four categories address general dimensions of teacher knowledge that were the mainstay of teacher education programs at the time. They were not the main focus of Shulman's work. Instead, they functioned as placeholders in a broader conception of teacher knowledge that emphasized con- tent knowledge. At the same time, however, Shulman made clear that these general categories were crucial and that an emphasis placed on content dimensions of teacher knowledge was not intended to minimize the importance of pedagogical understanding and skill: Shulman (1986) argued that "mere content knowledge is likely to be as useless pedagogically as content-free skill" (p. 8).

The second category, curricular knowledge, is "represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (p. 10). In addition, Shulman pointed to two other dimensions of curricular knowledge that are important for teaching, aspects that he labeled lateral curriculum knowledge and vertical curriculum knowledge. Lateral knowledge relates knowledge of the curriculum being taught to the curriculum that students are learning in other classes (in other subject areas). Vertical knowledge includes "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Shulman, 1986, p. 10).

The last, and arguably most influential, of the three contentrelated categories was the new concept of peda- gogical content knowledge. Shulman (1986) defined pedagogical content knowledge as comprising:

The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or diffi- cult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

The claim for pedagogical content knowledge was founded on observations that effective teachers in the Knowledge Growth in Teaching study represented key ideas using metaphors, diagrams, and explanations that were at once attuned to students' learning and to the integrity of the subject matter (see also Carlsen, 1988; Grossman, 1990; Marks, 1990; Wilson, 1988; Wilson et al., 1987; Wineburg, 1990). Some representations are especially powerful; others, although technically correct, do not open the ideas effectively to learners.

A second important idea is that representations of the subject are informed by content-specific knowledge of student conceptions. A focus on conceptions, and in many cases a particular interest in student misconcep- tions, acknowledges that accounting for how students understand a content domain is a key feature of the work of teaching that content. Grossman (1990) pointed out that these ideas are inherent in Dewey's admonition that teachers must learn to "psychologize" their subject matter for teaching, to rethink disciplinary topics to make them more acces- sible to students. Teachers must draw upon both their knowledge of subject matter to select appropriate topics and their knowledge of students' prior knowledge and conceptions to formulate appropriate and provocative representations of the content to be learned. (p. 8).

As a concept, pedagogical content knowledge, with its focus on representations and conceptions/misconcep- tions, broadened ideas about how knowledge might mat ter to teaching, suggesting that it is not only knowledge of content, on the one hand, and knowledge of pedagogy, on the other hand, but also a kind of amalgam of knowl- edge of content and pedagogy that is central to the knowledge needed for teaching. In Shulman's (1987) words, "Pedagogical content knowledge is the category most likely to distinguish the understanding of the con- tent specialist from the pedagogue" (p. 8).

Testing Shulman's Hypothesis About Content Knowledge and Pedagogical Content Knowledge

There was immediate and widespread interest in the ideas presented by Shulman and his colleagues. In the two decades since these ideas were first presented, Shulman's presidential address (1986) and the related Harvard Education Review article (1987) have been cited in more than 1,200 refereed journal articles. This interest has been sustained with no less than 50 citations to these two articles in every year since 1990. Perhaps most remarkable is the reach of this work, with citations appearing in 125 different journals, in professions rang- ing from law to nursing to business, and regarding knowledge for teaching students preschool through doc- toral studies. Much of the interest has focused directly on pedagogical content knowledge. Thousands of articles, book chapters, and reports use or claim to study the notion of pedagogical content knowledge, in a wide vari- ety of subject areas: science, mathematics, social studies, English, physical education, communication, religion, chemistry, engineering, music, special education, English language learning, higher education, and others. Such studies show no signs of abating. Rarely does an idea catch on so widely.

But how has the field taken up the idea of pedagogical content knowledge? What have we Much of the work that followed in the wake of Shulman's proposals showed how teachers' orientations to content influenced the ways in which they taught that con- tent. Grossman (1990) showed how teachers' orientations to literature shaped the ways in which they approached texts with their students. Wilson and Wineburg (1988) described how social studies teachers' disciplinary back- groundsin political science, anthropology, sociology shaped the ways in which they represented historical knowledge for high school students. And Ball (1990) introduced the phrase "knowledge about mathematics" to contrast with "knowledge of mathematics" and to high- light the nature of knowledge in the disciplinewhere it comes from, how it changes, and how truth is established. In science education, study of the "nature of science" showed that specific orientations are aligned with distinct subdisciplines and significantly influence the teaching carried out in classrooms. For instance, teachers trained in biology teach physics courses differently than do teachers trained in physics or in chemistry.

A second line of work some of it predating the introduction of pedagogical content knowledge has contributed to our understanding of the knowledge teachers need about common conceptions and miscon- ceptions that students bring to the classroom or develop as they learn a subject. For instance, Wineburg's (1990) analysis of students' natural efforts to understand motives and explanations for past events can be at cross-purposes with the special nature of historical understanding. Smith and Anderson (1984) showed that children's conceptions of food and eating persistently interfered with their learning about the process of photosynthesis as the means by which plants make their own food. Likewise, in the Cognitively Guided Instruction project, researchers found that students overgeneralize from experiences with problems in which the equals sign acts as a signal to compute (as it does in many programming languages) (Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000).

The notion of pedagogical content knowledge has permeated scholarship on teaching and teacher education but has done so unevenly across fields. Interestingly, our survey of the literature shows that roughly one fourth of the articles about pedagogical content knowledge are in science education, with slightly fewer in mathematics education. However, it is the breadth of literature on ped- agogical content knowledge that highlights the term's heuristic value as a way of conceptualizing teacher knowledge. In physical education, the term helps to dis- tinguish a teacher's own proficiency in a skill area (e.g., throwing a ball or dribbling) from the explicit knowledge of the skill that is needed in order to teach it to students (Chen, 2002; Rovegno, Chen, & Todorovich, 2003). There is a growing recognition that teaching reading requires a detailed knowledge of text, language, and read- ing process that goes beyond just being able to decode and comprehend text proficiently (Hapgood, Palincsar, Kucan, Gelpi-Lomangino, & Khasnabis, 2005; Moats, 1999; Phelps, 2005; Phelps & Schilling, 2004).

Still, however, the field has made little progress on Shulman's initial charge: to develop a coherent theoreti- cal framework for content knowledge for teaching. The ideas remain theoretically scattered, lacking clear defin- ition. Because researchers tend to specialize in a single subject, much of the work has unfolded in roughly par- allel but independent strands. Often it is unclear how ideas in one subject area relate to another or even whether findings within the same subject take similar or different views of teacher subject matter knowledge. Somewhat ironically, nearly one third of the articles that cite pedagogical content knowledge do so without direct attention to a specific content area, instead making general claims about teacher knowledge, teacher education, or policy. Scholars have used the concept of pedagogical content knowledge as though its theoretical foundations, conceptual distinctions, and empirical testing were already well defined and universally understood.

Particularly striking is the lack of definition of key terms. Pedagogical content knowledge is often not clearly distinguished from other forms of teacher knowl edge, sometimes referring to something that is simply content knowledge and sometimes to something that is largely pedagogical skill. Most definitions are perfunc- tory and often broadly conceived. This appears to be the case across all subject areas. For example, pedagogical content knowledge has been defined as "the intersection of knowledge of the subject with knowledge of teaching and learning" (Niess, 2005, p. 510) or as "that domain of teachers' knowledge that combines subject matter knowledge and knowledge of pedagogy" (Lowery, 2002, p. 69). In even broader terms, pedagogical content knowledge is defined simply as "the product of trans- forming subject matter into a form that will facilitate student learning" (de Berg & Greive, 1999, p. 20). Although these and a host of other short definitions capture the general idea of pedagogical content knowledge as a domain that combines the subject with teaching, they are broad enough to include nearly any package of teacher knowledge and beliefs.

Our Approach to Studying Mathematical Knowledge for Teaching

In the past, a focus on what teachers need to know has led to a set of positions, each related to principled argu- ments about what teachers should know. The prevailing view is that teachers need to know whatever mathemat- ics is in the curriculum plus some additional number of years of further study in college mathematics. A second hypothesis is that teachers need to know the curriculum, but "deeper," plus some amount of pedagogical content knowledge. In both cases, it is unclear what exactly it is that makes up the extra knowledge of mathematics.

A more focused question is this: What do teachers need to know and be able to do in order to teach effec- tively? Or, what does effective teaching require in terms of content understanding? This places the emphasis on the use of knowledge in and for teaching rather than on teachers themselves. These are centrally important ques- tions that could be investigated in numerous ways by examining the curriculum and standards for which teachers are responsible (or the tests their students must be prepared to pass), by asking expert mathematicians and mathematics educators to identify the core mathe- matical ideas and skills that teachers should have (CBMS, 2001), or by reviewing research on students' learning to ascertain those aspects of mathematics with which learners have difficulty (Stylianides & Ball, 2004). Our research group chose a different approach, one that might be characterized as working from the bot- tom up, beginning with practice. Because it seemed obvious that teachers need to know the topics and proce- dures that they teachprimes, equivalent fractions, functions, translations and rotations, factoring, and so on we decided to focus on how teachers need to know that content. In addition, we wanted to determine what else teachers need to know about mathematics and how and where teachers might use such mathematical knowl edge in practice.

Our analyses lay the foundation for a practice-based theory of mathematical knowledge for teaching (Ball & Bass, 2003b). We see this approach as a kind of job analy- sis, similar to analyses done of other mathematically intensive occupations that range from nursing, banking, and engineering (Hoyles, Noss, & Pozzi, 2001; Kent, Noss, Guile, Hoyles, & Bakker , 2007; Noss & Hoyles, 1996) to carpentry and waiting tables (Milroy, 1992).

METHODOLOGY

Our study of the mathematical demands of teaching has yielded a wealth of tasks that require mathematical knowledge and skill. What caught us by surprise, however, was how much special mathematical knowledge was required, even in many everyday tasks of teaching assigning student work, listening to student talk, grading or commenting on student work. Despite the fact that these tasks are done with and for students, close analysis revealed how intensively mathematical the tasks were. We were surprised to see that many of the component tasks of teaching require mathematical knowledge apart from knowledge of students or teaching. For instance, deciding whether a method or procedure would work in general requires mathematical knowledge and skill, not knowledge of students or teaching. It is a form of mathematical problem solving used in the work of teaching. Likewise, determining the validity of a mathematical argument, or selecting a mathematically appropriate rep- resentation, requires mathematical knowledge and skill important for teaching yet not entailing knowledge of students or teaching. In our research we began to notice how rarely these mathematical demands could be addressed with mathematical knowledge learned in university mathematics courses. We began to hypothesize that there were aspects of subject matter knowledgein addition to pedagogical content knowledge that need to be uncovered, mapped, organized, and included in math ematics courses for teachers.

Three points are central to our argument. First, much of the work of teaching is mathematical in nature, with significant mathematical demands. Although the mathematical tasks we have identified would inform teachers' choices and actions with students, these tasks can also be seen as illustrating the special mathematical thinking that teachers must do and understand in order to teach mathematics. These tasks require significant mathematical knowledge, skill, habits of mind, and insight. Although our examples are drawn from the context of teaching, the mathematical knowledge needed to engage them stands on its own as a domain of understanding, disposition, and skill needed by teachers for their work. A second point is that the mathematical knowledge we have identified here has a relevance to teaching that is often missing from discussions about the mathematics needed by teachers. By identifying mathematics in rela- tion to specific tasks in which teachers engage, we establish its relevance to what teachers do. Part of the value of the notion of pedagogical content knowledge is that it offers a way to build bridges between the academic world of disciplinary knowledge and the practice world of teaching; it does so by identifying amalgam knowledge that combines the knowing of content with the knowing of students and pedagogy. Our practice-based conceptualization of content knowledge for teaching provides an additional way of building bridges between these two worlds; it does so by defining knowledge in broad terms, including skill, habits of mind, and insight, and by framing knowledge in terms of its use in terms of particular tasks of teaching.

Finally, we suspect that many of these insights extend to the knowledge teachers need in other subjects as well. What might these insights mean in the context of teaching history, or biology, or music? In our analyses of the mathematical work involved in teaching mathematics, we noticed that the nature of that mathematical knowledge and skill seemed itself to be of different types. We hypothesized that teachers' opportu- nities to learn mathematics for teaching could be better tuned if we could identify those types more clearly. If mathematical knowledge required for teaching is indeed multidimensional, then professional education could be organized to help teachers learn the range of knowledge and skill they need in focused ways. If, however, the mathematical knowledge required for teaching is basically the same as general mathematical ability, then dis- criminating professional learning opportunities would be unnecessary. Based on our analysis of the mathematical demands of teaching, we hypothesized that Shulman's content knowledge could be subdivided into CCK and specialized content knowledge and his pedagogical con- tent knowledge could be divided into knowledge of con- tent and students and knowledge of content and teaching. Turning back to the results of our studies, in the next sec- tion we define and illustrate each of these subdomains.

RESEARCH RESULT

Mathematical Knowledge for Teaching and Its Structure

When teachers write on the board, they need to use terms and notation correctly. In short, they must to be able to do the work that they assign their students. But some of this requires mathematical knowledge and skill that others have as well thus, it is not special to the work of teaching. By "common," however, we do not mean to suggest that everyone has this knowledge. Rather, we mean to indicate that this is knowledge of a kind used in a wide variety of settings in other words, not unique to teaching.

When we analyzed videos of teaching, it was obvious that such knowledge is essential. When a teacher mispronounced terms, made calculation errors, or got stuck trying to solve a problem at the board, instruction suffered and valuable time was lost. In mapping out the mathematical knowledge needed by teachers, we found that an understanding of the mathematics in the student curriculum plays a critical role in planning and carrying out instruction. Additional evidence for common content knowledge comes from our work to develop instruments for measuring mathematical knowledge for teaching. We pose questions such as, "What is a number that lies between 1.1 and 1.11?" We ask questions that require knowing that a square is a rectangle, that 0/7 is 0, and that the diago- nals of a parallelogram are not necessarily perpendicular. These are not specialized understandings but are ques- tions that typically would be answerable by others who know mathematics. Often, as shown in Figure 2, we couch the problem in the context of teaching to point out where in the activity of teaching the use of such common knowledge might arise. In analyzing the mathematical demands of teaching, we seek to identify mathematical knowledge that is demanded by the work teachers do. To pursue this, we define the mathematical knowledge we are studying as mathematical knowledge "entailed by teaching" in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students. To avoid a strictly reductionist and utilitarian perspective, however, we seek a generous conception of "need" that allows for the per- spective, habits of mind, and appreciation that matter for effective teaching of the discipline. The first domain represents the first step in th example above: simply calculating an answer or, more generally, correctly solving mathematics problems. We Presenting mathematical ideas Responding to students' "why" questions Finding an example to make a specific mathematical point Recognizing what is involved in using a particular representation Linking representations to underlying ideas and to other representations Connecting a topic being taught to topics from prior or future years Explaining mathematical goals and purposes to parents Appraising and adapting the mathematical content of textbooks Modifying tasks to be either easier or harder Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book did. She came across a page that asked students to determine if a few statements about 0 were true or f alse. Which statement(s) should she recognize as true?

	Yes	No
a) 0 is an even number.	1	2
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2

call this common content knowledge (CCK) and define it as the mathematical knowledge and skill used in settings other than teaching. Teachers need to know the material they teach; they must recognize when their students give wrong answers or when the textbook gives an inaccurate

The mathematics of this problem can be rather challenging. The first word problem is division by 2 rather than by ½; the second is multiplication by 2 rather than division by ½ (a subtle yet important point for teaching this content); and the third correctly fits the calculation using a measurement meaning of division. The important point here, though, is that figuring out which story prob- lems fit with which calculations, and vice versa, is a task engaged in teaching this content, not something done in solving problems with this content.

Teaching involves the use of decompressed mathe- matical knowledge that might be taught directly to students as they develop understanding. However, with students the goal is to develop fluency with compressed mathematical knowledge. In the end, learners should be able to use sophisticated mathematical ideas and proce- dures. Teachers, however, must hold unpacked mathe- matical knowledge because teaching involves making features of particular content visible to and learnable by students. Teaching about place value, for example, requires understanding the place-value system in a self- conscious way that goes beyond the kind of tacit under- standing of place value needed by most people. Teachers, however, must be able to talk explicitly about how math- ematical language is used (e.g., how the mathematical meaning of edge is different from the everyday reference to the edge of a table); how to choose, make, and use mathematical representations effectively (e.g., recogniz- ing advantages and disadvantages of using rectangles or circles to compare fractions); and how to explain and justify one's mathematical ideas (e.g., why you invert and multiply to divide fractions). All of these are examples of ways in which teachers work with mathe- matics in its decompressed or unpacked form.

Some might wonder whether this decompressed knowledge is equivalent to conceptual understanding. They might ask whether we would not want all learners to understand content in such ways. Our answer is no. What we are describing is more than a solid grasp of the material. We do not hold as a goal that every learner should be able to select examples with pedagogically strategic intent, to identify and distinguish the complete range of different situations modeled by 38 ÷ 4, or to analyze common errors.

The mathematical demands of teaching require spe- cialized mathematical knowledge not needed in other settings. Accountants have to calculate and reconcile numbers and engineers have to mathematically model properties of materials, but neither group needs to explain why, when you multiply by 10, you "add a zero." In developing survey questions to measure such knowl- edge, we ask, for example, whether an unusual method proposed by a student would work in general, which statement best explains why we find common denomina- tors when adding fractions, and which of a set of given drawings could be used to represent 2 divided by 2/3. These and questions like them are the daily fare of teach- ing. The demands of the work of teaching mathematics create the need for such a body of mathematical knowl- edge specialized to teaching.

The third domain, knowledge of content and students (KCS), is knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interest- ing and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and famil iarity with students and their mathematical thinking.

Central to these tasks is knowledge of common student conceptions and misconceptions about particular mathe- matical content. For instance, in the subtraction example, knowing that students often "subtract up" when confronted with a problem such as 307-168 means that a teacher who has seen this happen and knows that it is a common student response is able to recognize it without extensive mathe- matical analysis or probing. In other words, recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowl- edge of content and students (KCS).

Many demands of teaching require knowledge at the intersection of content and students. In developing an instrument to measure such knowledge, we ask ques- tions, for example, about the kinds of shapes young students are likely to identify as triangles, the likelihood that they may write 405 for 45, and problems where con- fusion between area and perimeter lead to erroneous answers. We also ask questions that require interpreta- tion of students' emerging and inchoate thinking, that present the thinking or expressions typical of particular learners, or that demand sensitivity to what is likely to be easy or challenging.

Many of our ideas in this area draw from the literature on student thinking: for example, van Hiele's studies of levels of the development in representing two-dimensional figures (Burger & Shaughnessy, 1986; Crowley, 1987), CGI researchers' documentation of common misinterpretations of the equal sign (Carpenter et al., 2003; Carpenter & Levi, 2000) or that subtraction problems involving comparison are harder for students than take-away problems (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998), or Philipp, Cabral, and Schappelle's (2005) observation that students misappropriate the subtraction language of take-away when representing fractions, causing them to confound what is left with what is removed. In each case, knowledge of students and content is an amalgam, involv- ing a particular mathematical idea or procedure and famil iarity with what students often think or do.

The last domain, knowledge of content and teaching (KCT), combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. Teachers sequence particular content for instruction. They choose which examples to start with and which examples to use to take students deeper into the content. Teachers evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning.

Consider for a moment the need to make instructional decisions about which student contributions to pursue and which to ignore or save for a later time. During a classroom discussion, a teacher must decide when to pause for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning. Each of these decisions requires coordination between the mathematics at stake and the instructional options and purposes at play. One example of KCT would be knowing different instructionally viable models for place value, knowing what each can reveal about the subtraction algorithm, and knowing how to deploy them effectively. What does money afford instructionally for a particular subtraction problem and how is this different from what coffee stir- rers bundled with rubber bands would afford? What about base-ten blocks or "unifix" cubes? Each of these can cor- rectly represent subtraction of multidigit numbers, but each represents different aspects of the content that make a difference at different points in students' learning. Each model also requires different care in use in order to make the mathematical issues salient and usable by students (Cohen, 2005). Knowing how these differences matter for the development of the topic is part of what we call knowledge of content and teaching.

The demands of teaching require knowledge at the intersection of content and teaching. In developing an instrument to measure such knowledge, we ask questions about whether a tape measure would be good for teaching place value, about choosing examples for simplifying radicals for the purpose of discussing multiple strategies, or about sequencing subtraction problems with and with- out regrouping for instruction. We also ask questions about how language and metaphors can assist and con- found student learning the way language about bor- rowing or canceling may interfere with understanding of the mathematical principles underlying the subtraction algorithm or the solving of algebraic equations. In each of these examples, knowledge of teaching and content is an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content.

DISCUSSION

Building a Map of Usable Professional Knowledge of Subject Matter

Several issues about our proposed categories are worth addressing their relationship to pedagogical con- tent knowledge, the special nature of specialized content knowledge, our use of teaching as a basis for defining the domains, and problems with the categories that need to be addressed.

From our definitions and examples it should be evident that this work may be understood as elaborating on, not replacing, the construct of pedagogical content knowledge. For instance, the last two domains knowledge of content and students and knowledge of content and teaching coincide with the two central dimensions of pedagogical content knowledge identified by Shulman (1986): "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught top- ics and lessons" and "the ways of representing and for mulating the subject that make it comprehensible to others" (p. 9).

However, we also see our work as developing in more detail the fundamentals of subject matter knowledge for teaching by establishing a practice-based conceptualiza tion of it, by elaborating subdomains, and by measuring and validating knowledge of those domains.

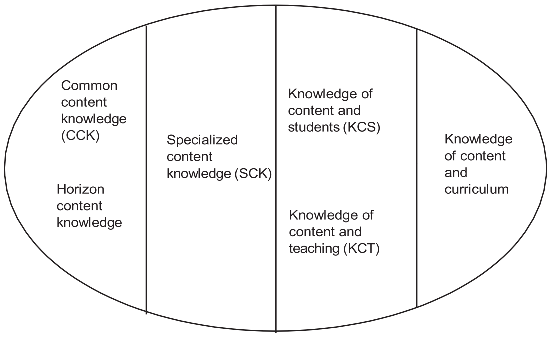
We have been most struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject that is not intertwined with knowledge of pedagogy, students, curriculum, or other noncontent domains. What distinguishes this sort of mathematical knowledge from other knowledge of mathematics is that it is subject matter knowledge needed by teachers for specific tasks of teaching, such as those in Figure 3, but still clearly subject matter knowledge. These tasks of teaching depend on mathematical knowledge, and, sig nificantly, they have aspects that do not depend on knowledge of students or of teaching. These tasks require knowing how knowledge is generated and structured in the discipline and how such considerations mat ter in teaching, such as extending procedures and concepts of whole-number computation to the context of rational numbers in ways that preserve properties and meaning. These tasks also require a host of other mathematical knowledge and skills knowledge and skills not typically taught to teachers in the course of their formal mathematics preparation.

Where for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Where do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime or how and why the long division algorithm works? Teachers must know these sorts of things and engage in these mathematical practices them selves when teaching. Explicit knowledge and skills in these areas are vital for teaching.

To represent our current hypotheses, we propose a dia gram as a refinement to Shulman's categories. Figure 5 shows the correspondence between our current map of the domain of content knowledge for teaching and two of Shulman's (1986) initial categories: subject matter knowledge and pedagogical content knowledge. We have provisionally placed Shulman's third category, curricular knowledge, within pedagogical content knowledge. This is consistent with later publications from members of.

Figure 5
Domains of Mathematical Knowledge for Teaching

Subject Matter Knowledge Pedagogical Content Knowledge



Shulman's research team (Grossman, 1990). We are not yet sure whether this may be a part of our category of knowledge of content and teaching or whether it may run across the several categories or be a category in its own right. We also provisionally include a third category within subject matter knowledge, what we call "horizon knowledge" (Ball, 1993). Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First- grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later. It also includes the vision useful in seeing connections to much later mathematical ideas. Having this sort of knowledge of the mathematical horizon can help in making decisions about how, for example, to talk about the number line. Likewise third graders appreciate that the number line they know will soon "fill in" with more and more numbers? And might it matter how a teacher's choices anticipate or distort that later development? Again we are not sure whether this category is part of subject matter knowledge or whether it may run across the other cate- gories. We hope to explore these ideas theoretically, empirically, and also pragmatically as the ideas are used in teacher education or in the development of curriculum materials for use in professional development.

Our current empirical results, based on our factor analyses, suggest it is likely that content knowledge for teaching is multidimensional (Hill et al., 2004; Schilling, in press). Whether these categories, as we propose them here, are the right ones is not most important. Likely they are not. Our current categories will continue to need.

CONCLUSION

Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher com- petency. The reason is simple: Teachers who do not themselves know a subject well are not likely to have the knowledge they need to help students learn this content. At the same time, however, just knowing a subject well may not be sufficient for teaching. One need only sit in a classroom for a few minutes to notice that the mathe particular representation is specialized knowledge; it is hard to think of others who use this knowledge in their day-to-day work. But perhaps there are others who rely on such detailed and unpacked knowledge of fractions in their work as well. Similarly, it can be difficult at times to discriminate specialized content knowledge from knowledge of content and students––for example, con- sider what is involved in selecting a numerical example to investigate students' understanding of decimal numbers. The shifts that occur across the four domains, for example, ordering a list of decimals (CCK), generat- ing a list to be ordered that would reveal key mathemat- ical issues (SCK), recognizing which decimals would cause students the most difficulty (KCS), and deciding what to do about their difficulties (KCT), are important yet subtle. That we are able to work empirically as well as conceptually helps us to refine our categories; still, we recognize the problems of definition and precision exhibited by our current formulation.

Finally, we need to understand better the extent to which our formulation of mathematical knowledge for teaching is culturally specific (Cole, 2008; Delaney, 2008) or dependent on teaching styles. We do not think of the knowledge we have been identifying as being closely tied to a particular view of reform or a particular approach to teaching. For instance, interpreting students' thinking, whether in a whole-class discussion or on writ ten homework or a quiz, is an essential part of effectively engaging students in the learning of subject matter. Explaining mathematical ideas is central to teaching, whatever the approach or style. Writing assessment questions, drawing a clear diagram, choosing a counterexampleeach of these is a core task of teaching. Still, although our analyses are designed to consider fun damental tasks of teaching content, the particular sample of data we use clearly influences what we do and do not see, and the question of its limitations remains an empir- ical question.

First, in studying the relationships between teachers' content knowledge and their students' achievement, it would be useful to ascertain whether there are aspects of teachers content knowledge that predict student achieve- ment more than others. If, for instance, teachers' special ized content knowledge is the greatest predictor of students' achievement, this might direct our efforts in ways different than if advanced content knowledge has the largest effect. However, such studies are sorely miss- ing. Second, it could be useful to study whether and how different approaches to teacher development have different effects on particular aspects of teachers' pedagogical content knowledge. Third, and closely related, a clearer sense of the categories of content knowledge for teaching might inform the design of support materials for teachers as well as teacher education and professional development. Indeed, it might clarify a curriculum for the content preparation of teachers that is professionally based both distinctive, substantial and fundamentally tied to professional practice and to the knowledge and skill demanded by the work.

The work reported here takes Shulman's charge seriously. It is rooted in attention to the demands of practice to consider what mathematics arises in the work that teachers do. Our work tests these ideas by developing instruments to measure this knowledge, by using the results to inform our understanding of a map of teacher content knowledge, and by tying this knowledge to its use in practice. That there is a domain of content knowledge unique to the work of teaching is a hypothesis that has already developed. However, the notion of specialized content knowledge is in need of further work in order to understand the most important dimensions of teachers' professional knowledge. Doing so with care promises to have significant implications for understanding teaching and for improving the content preparation of teachers.
sense of student work and choosing powerful ways of representing the subject so that it is understandable to students. It seems unlikely that just knowing more advanced math will satisfy all of the content demands of teaching. In fact,

elementary teachers' mathematics course attainment does not predict their students' achievement gains (National Mathematics Advisory Panel, 2008). What seem most important are knowing and being able to use the mathematics required inside the work of teaching.

Unfortunately, subject matter courses in teacher preparation programs tend to be academic in both the best and worst sense of the word, scholarly and irrele- vant, either way remote from classroom teaching. Disciplinary knowledge has the tendency to be oriented in directions other than teaching, toward the discipline history courses toward knowledge and methods for doing history and science courses toward knowledge and methods for doing science. Although there are excep- tions, the overwhelming majority of subject matter courses for teachers, and teacher education courses in general, are viewed by teachers, policy makers, and society at large as having little bearing on the day-to-day realities of teaching and little effect on the improvement of teaching and learning. This is the problem that Shulman and his colleagues addressed in the late 1980s. In this article, we argue that the issues identified by Shulman and his colleagues more than two decades ago are key to research on teaching and teacher education. Content knowledge is immensely important to teaching and its improvement. Instead of taking pedagogical con- tent knowledge as given, however, we argue that there is a need to carefully map it and measure it. This includes the need to better explicate how this knowledge is used in teaching effectively.

Why are new categories useful? Three reasons capture our current thinking about the usefulness of refining the conceptual map of the content knowledge for teachers.

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